INDETERMINACY IN A LOG-LINEARIZED NEOCLASSICAL GROWTH MODEL WITH QUASI-GEOMETRIC DISCOUNTING*

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ABSTRACT

This paper studies the properties of solutions to a log-linearized version of the neoclassical growth model with quasi-geometric discounting. We show that after the log-linearization, the model has indeterminacy and multiplicity of equilibria even though the original non-linear model has a unique interior solution. Specifically, in both the deterministic and stochastic cases, the log-linearized model has a continuum of steady states. In the deterministic case, there is a unique log-linear policy function leading to each steady state, while in the stochastic case, there is a continuum of log-linear policy functions, associated with each steady state. Hence, the standard log-linearization method cannot be applied for solving models with quasi-geometric discounting.

Keywords: quasi-geometric (quasi-hyperbolic) discounting, time inconsistency, neoclassical growth model, indeterminacy, multiplicity, numerical methods

JEL Classification: C73, D90, E21

1 Introduction

Under the assumption of quasi-geometric (quasi-hyperbolic) discounting, the consumer's short-run discount factor is different from the long-run discount factor. If the short-run discount factor is lower than the long-run discount factor, then the consumer is short-run impatient: she always plans to save much in the next period, however, as the next period comes around, she delays savings to the future. On the contrary, if the short-run discount factor is higher than the long-run one, the consumer is short-run patient and always saves more than she has originally planned.[1]

It has been shown in the literature that the assumption of quasi-geometric discounting leads to the indeterminacy and multiplicity of equilibria. Krusell and Smith (2002) study the neoclassical growth model with quasi-geometric discounting, which has a closed-form solution, and report that "...there are fundamental problems in finding algorithms that succeed in producing accurate solutions, at least when the individual-specific uncertainty is limited". They show that, in addition to a smooth closed-form solution, the model has infinitely many discontinuous solutions: there are both a continuum of steady states and a continuum of decision rules associated with each steady state. The constructed decision rules have the form of step functions such that the propensity to save is equal to zero in all points except of those, in which the steps are taken. Krusell and Smith (2002) conclude: "The results herein suggest an explanation for the numerical problems: the lack of convergence of algorithms appears to be cycling within the large set of equilibria".

Krusell, Kuruşçu and Smith (2001) and Maliar and Maliar (2002) argue that it is possible to rule out the indeterminacy and multiplicity by restricting attention to an interior equilibrium (the one that satisfies the Euler equation). We begin by showing the following related result: If we restrict the equilibrium to be interior on the whole domain of capital, the problem of finding the optimal policy and value functions in the model with the closed-form solution, studied in Krusell and Smith (2000), is a contraction mapping. In such a model, starting from an arbitrary initial guess on value function, the sequence of value functions computed iteratively converges to the closed-form solution. The same is true for the stochastic setting where the logarithm of

[1] The recent literature that studies models with quasi-geometric discounting includes, e.g., Laibson (1997), Repetto, Tobacman and Laibson (1998), Barro (1999), Harris and Laibson (1999), Caillaud and Jullien (2000), Krusell and Smith (2000), Krusell, Kuruşçu and Smith (2001), and Maliar and Maliar (2002).

technology follows an $AR(1)$ process. This example suggests that it is of interest to investigate the properties of the solutions to the Euler equation.

In the present paper, we focus exclusively on the performance of the log-linearization method, which belongs to the class of Euler-equation-based methods. Our findings are as follows: Even though the original non-linear model has a unique interior solution, the log-linearized version of the model has the indeterminacy of both the steady state and the near-steady-state dynamics. Specifically, there is an interval, each point of which can be a steady state.[2] In the deterministic case, there is a unique log-linear policy function leading to each particular steady state.[3] In the stochastic case, the indeterminacy is even more severe: there is a continuum of log-linear policy functions, associated with each steady state. Hence, we conclude that the standard log-linearization method cannot be used for solving models with quasi-geometric discounting.

The rest of the paper is as follows: Section 2 formulates the neoclassical growth model with quasi-geometric discounting. Section 3 describes the recursive formulation and derives the Euler equation. Section 4 discusses the setup with a closed-form solution. Section 5 derives a solution to a log-linearized version of the model. Section 6 studies the optimality of the constructed log-linear policy rules. Section 7 discusses the results. Finally, Section 8 concludes.

2 The model

Time is discrete and infinite, $t \in \{0, 1, 2, ...\}$. On every date t, an agent chooses a stochastic sequence of consumption $\left\{c_t^t, c_{t+1}^t, c_{t+2}^t, ...\right\}$, where the time superscript and time subscript indicate periods in which and for which consumption is chosen, respectively, (e.g., consumption c_{t+1}^t is chosen in period t for period $t+1$). The agent's preferences in period t are

$$U\left(c_t^t, c_{t+1}^t, c_{t+2}^t, ...\right) = u\left(c_t^t\right) + \beta\delta E_t u\left(c_{t+1}^t\right) + \beta\delta^2 E_t u\left(c_{t+2}^t\right) + ...,$$

[2] Our multiplicity interval for the log-linear solutions coincides with the one obtained in Krusell and Smith (2000) for the step-function equilibria.

[3] Regarding step function equilibria, in the deterministic version of the model, Krusell and Smith (2000) show that, in a neighborhood of each steady state, there exists a continuum of step functions leading to a given steady state.

where E_t is denotes the conditional expectation, $\beta > 0$ and $\delta \in (0,1)$ are discounting parameters. We assume that the period utility function $u(c)$ is continuously differentiable, strictly increasing and strictly concave.

The weights on the period utilities from period t forward are given by 1, $\beta\delta \cdot 1$, $\delta \cdot \beta\delta$, $\delta \cdot \beta\delta^2$, ... Krusell and Smith (2000) call such discounting quasi-geometric because with an exception of the current period t, the weights decline geometrically over time. The standard case of geometric discounting corresponds to $\beta = 1$. If $\beta > 1$ ($\beta < 1$), then the short-run discount factor, $\beta\delta$, is higher (lower) than the long-run one, δ, so that an agent is short-run patient (impatient). The case of $\beta < 1$ is also referred to in the literature as quasi-hyperbolic discounting, (see, e.g., Laibson, 1997, Harris and Laibson, 1999).

The assumption of quasi-geometric discounting leads to time-inconsistency in preferences in the sense that the relative value of consumption in any two adjacent periods t and $t+1$ depends on the date at which the evaluation is performed. Specifically, at time $t-1$, one unit of c_{t+1}^{t-1} is valued δ times less than one unit of c_t^{t-1}, while at time t, one unit of c_{t+1}^t gives $\beta\delta$ times less utility than one unit of c_t^t. We suppose that the agent is fully aware of her preference inconsistency.

The agent runs a production technology $\theta_t f(k_t)$, where θ_t is a technology shock, and k_t is the capital stock. Therefore, on each date t, the agent solves the following utility maximization problem

$$\max_{\left\{c_\tau^t, k_{\tau+1}^t\right\}_{\tau=t}^\infty} \left\{ u\left(c_t^t\right) + E_t \sum_{\tau=t}^\infty \beta\delta^{\tau+1-t} u\left(c_{\tau+1}^t\right) \right\} \tag{1}$$

$$\text{s.t.} \qquad c_\tau^t + k_{\tau+1}^t = (1-d)\, k_\tau^{t-1} + \theta_\tau f\left(k_\tau^{t-1}\right), \tag{2}$$

where $d \in (0,1]$ is the depreciation rate of capital. The function f is strictly increasing, strictly concave, continuously differentiable and satisfies the Inada conditions. We assume that the random variable $\ln\theta_{t+1}$ follows the $AR(1)$ process: $\ln\theta_{t+1} = \rho\ln\theta_t + \varepsilon_{t+1}$, where $\rho \in [0,1)$ and $\varepsilon_{t+1} \sim N(0, \sigma^2)$.

Due to time-inconsistency, consumption considered to be optimal at t, c_{t+1}^t, is not equal to the one chosen at $t+1$, c_{t+1}^{t+1}. The "true" consumption at $t+1$ is c_{t+1}^{t+1}. Therefore, the "true" lifetime stream of consumption is $\{c_0^0, c_1^1, ...\} \equiv \{c_0, c_1, ...\}$. Similarly, the "true" sequence of capital is given by $\{k_1^0, k_2^1, ...\} \equiv \{k_1, k_2, ...\}$. We assume that the agent cannot commit to her future actions. If commitment was possible at any time t, then a sequence $\left\{c_t^t, c_{t+1}^t, c_{t+2}^t, ...\right\}$ solving (1), (2) at t would be the "true" one.

3 Recursive formulation and Euler equation

The problem $(1), (2)$ can be written recursively. A recursive (Markov) equilibrium is defined as one, in which the agent chooses the next period capital stock k_{t+1} according to a time-invariant policy function, $k_{t+1} = g(k_t, \theta_t)$. We use $W(k_t, \theta_t)$ to denote the optimal value of the expected discounted utility of the agent whose current state is k_t and θ_t, and who starting from period $t+1$ and forward, makes her decisions according to the policy function g. Without time subscripts, a recursive formulation is

$$W(k,\theta) = \max_{k'} \{u((1-d)k + \theta f(k) - k') + \beta\delta E[V(k',\theta') \mid \theta]\}, \tag{3}$$

where $V(k,\theta)$ satisfies the recursive functional equation

$$V(k,\theta) = u((1-d)k + \theta f(k) - g(k,\theta)) + \delta E\{V[g(k,\theta);\theta'] \mid \theta\}, \tag{4}$$

and k, θ are given. A solution to this problem is given by the optimal functions $W(k,\theta)$, $V(k,\theta)$ and $g(k,\theta)$.

We assume that equilibrium is interior. Then, one can derive the optimality condition of the problem $(3), (4)$. The first-order necessary condition with respect to k' is

$$u'(c) = \beta\delta E\left[\frac{\partial V(k',\theta')}{\partial k'}\right]. \tag{5}$$

The derivative of the value functions $V(k',\theta')$ with respect to k' is

$$\frac{\partial V(k',\theta')}{\partial k'} = u'(c')\left(1-d+\theta' f'(k') - \frac{\partial g(k',\theta')}{\partial k'}\right) + \frac{\partial g(k',\theta')}{\partial k'}\delta E\left[\frac{\partial V(k'',\theta'')}{\partial k''}\right], \tag{6}$$

where $\partial g / \partial k'$ is taken out of the expectation because it is known before the shock θ'' is realized. By updating (5) and substituting it into (6), we obtain

$$\beta\frac{\partial V(k',\theta')}{\partial k'} = u'(c')\left(\beta(1-d+\theta' f'(k')) + (1-\beta)\frac{\partial g(k',\theta')}{\partial k'}\right). \tag{7}$$

Conditions (5) and (7) together imply

$$u'(c_t) = \delta E_t\left\{u'(c_{t+1})\left(\beta(1-d+\theta_{t+1}f'(k_{t+1})) + (1-\beta)\frac{\partial g(k_{t+1},\theta_{t+1})}{\partial k_{t+1}}\right)\right\}. \tag{8}$$

This is the Euler equation in the case of quasi-geometric discounting. Note that the agent's consumption-saving decision at time t depends not only on the future return on capital but also on the future marginal propensity to save out of capital, $\frac{\partial g(k_{t+1},\theta_{t+1})}{\partial k_{t+1}}$. This feature of the model plays a determinant role in the properties of the solution.

4 Closed-form solution

We begin our analysis by considering a version of the model that admits a closed-form solution. Assume that the period utility function is logarithmic, $u(c) = \ln(c)$, that the production function is Cobb-Douglas, $f(k) = k^{\alpha}$, with $\alpha \in (0,1)$ and that capital depreciates fully during each period, $d = 1$. Assume also that the equilibrium is interior. Then, the value and the policy functions, solving (3), (4), are given by

$$\begin{aligned} V(k,\theta) &= (1-\delta)^{-1}\left(\ln\frac{1-\delta\alpha}{1-\delta\alpha+\beta\delta\alpha}+\frac{\delta\alpha}{1-\delta\alpha}\ln\frac{\beta\delta\alpha}{1-\delta\alpha+\beta\delta\alpha}\right) \\ &\quad +\frac{\alpha}{1-\delta\alpha}\ln k+\frac{1}{(1-\delta\rho)(1-\delta\alpha)}\ln\theta, \end{aligned} \tag{9}$$

$$k' = \frac{\beta\delta\alpha}{1-\delta\alpha+\beta\delta\alpha}\theta k^{\alpha}. \tag{10}$$

In the deterministic case, Krusell and Smith (2000) obtain closed-form solution (9), (10) by using the guess-and-verify method.

Krusell and Smith (2000) report that numerical algorithms iterating on value function fail to converge to the closed-form solution. They explain their finding by the fact that, in addition to the smooth closed-form solution, there are infinitely many discontinuous solutions in the form of step functions. Krusell, Kuruşçu and Smith (2001) and Maliar and Maliar (2002) argue that one can rule out the multiplicity by restricting attention to an interior equilibrium. In Appendix A, we illustrate the uniqueness of an interior equilibrium in the model with a closed-form solution by iterating on value function "by hand". To be specific, we show that starting from an initial guess $V = 0$, a sequence of value functions computed iteratively converges to value function (9). Hence, the problem of finding the policy and value functions solving (3), (4) is a contraction mapping, and closed-form solution (9), (10) is a unique limit of the solution to the finite-horizon problem.

Thus, a natural question arises: "Will numerical methods based on the Euler equation allow us to find an interior equilibrium?" In this paper, we study the performance of one well-known Euler equation method, which is the log-linearization.

5 A log-linearization method

In the paper, we concentrate exclusively on the dynamics produced by a log-linearized version of the model. We specifically log-linearize the Euler equation (8) and budget constraint (2) around a steady state and study the properties of the log-linear solutions. The assumption, which is clearly indispensable for our analysis, is that the model is consistent with the steady state.

We denote by $\overline{x}$ steady state value of a variable x_t, and by $\widehat{x}_t = \frac{x_t - \overline{x}}{\overline{x}}$ the log-deviation of the variable x_t from a steady state. In the steady state, the Euler equation (8) and budget constraint (2) are

$$1 = \delta \left(\beta \left(1 - d + \overline{\theta} f'\left(\overline{k}\right)\right) + (1-\beta) \frac{\partial g\left(\overline{k}, \overline{\theta}\right)}{\partial k} \right), \tag{11}$$

$$\overline{c} = \overline{\theta} f\left(\overline{k}\right) - \overline{k} d. \tag{12}$$

In this section, we assume that in the steady state $\overline{c}$, $\overline{k}$, $\overline{\theta}$, the policy function g is twice continuously differentiable, and we approximate its first-order derivative as $\frac{\partial g(k_t, \theta_t)}{\partial k_t} \simeq \frac{\partial g\left(\overline{k}, \overline{\theta}\right)}{\partial k} + \frac{\partial^2 g\left(\overline{k}, \overline{\theta}\right)}{\partial k^2} \cdot \overline{k}\, \widehat{k}_{t+1} + \frac{\partial^2 g\left(\overline{k}, \overline{\theta}\right)}{\partial k \partial \theta} \cdot \overline{\theta}\, \widehat{\theta}_{t+1}$. By log-linearizing the Euler equation (8) and budget constraint (2) around the steady state, we obtain

$$\begin{aligned} \frac{u''\left(\overline{c}\right)}{u'\left(\overline{c}\right)} \overline{c}\, \widehat{c}_t = \frac{u''\left(\overline{c}\right)}{u'\left(\overline{c}\right)} \overline{c}\, \widehat{c}_{t+1} + \delta \left(\beta f'\left(\overline{k}\right) + (1-\beta) \frac{\partial^2 g\left(\overline{k}, \overline{\theta}\right)}{\partial k \partial \theta} \right) \cdot \overline{\theta}\, \widehat{\theta}_{t+1} \\ + \delta \left(\beta \overline{\theta} f''\left(\overline{k}\right) + (1-\beta) \frac{\partial^2 g\left(\overline{k}, \overline{\theta}\right)}{\partial k^2} \right) \cdot \overline{k}\, \widehat{k}_{t+1}, \end{aligned} \tag{13}$$

$$\overline{c}\, \widehat{c}_t + \overline{k}\, \widehat{k}_{t+1} = f\left(\overline{k}\right) \cdot \overline{\theta}\, \widehat{\theta}_t + \left(1 - d + \overline{\theta} f'\left(\overline{k}\right)\right) \cdot \overline{k}\, \widehat{k}_t. \tag{14}$$

We characterize the near-steady-state dynamics by using the method of undetermined coefficients (see, e.g., Uhlig, 1999). We postulate

$$\widehat{k}_{t+1} = \xi_{kk}\widehat{k}_t + \xi_{k\theta}\widehat{\theta}_t, \qquad \widehat{c}_t = \xi_{ck}\widehat{k}_t + \xi_{c\theta}\widehat{\theta}_t, \tag{15}$$

where ξ_{kk}, $\xi_{k\theta}$, ξ_{ck} and $\xi_{c\theta}$ are the coefficients to be determined. Note that the policy rule for capital in (15) implies that $\frac{\partial g(\overline{k},\overline{\theta})}{\partial k} = \xi_{kk}$.

Equations (13), (14) and (15) imply the following four restrictions:

$$\frac{u''(\overline{c})}{u'(\overline{c})}\overline{c}\,\xi_{ck}(1-\xi_{kk}) + \delta\left(\beta\overline{\theta}f''(\overline{k}) + (1-\beta)\frac{\partial^2 g(\overline{k},\overline{\theta})}{\partial k^2}\right)\cdot\overline{k}\,\xi_{kk} = 0, \tag{16}$$

$$\begin{aligned}\frac{u''(\overline{c})}{u'(\overline{c})}\overline{c}\,(\xi_{c\theta} - \xi_{ck}\xi_{k\theta} - \xi_{c\theta}\rho) + \delta\rho\left(\beta f'(\overline{k}) + (1-\beta)\frac{\partial^2 g(\overline{k},\overline{\theta})}{\partial k\partial\theta}\right)\cdot\overline{\theta} + \\ + \delta\left(\beta\overline{\theta}f''(\overline{k}) + (1-\beta)\frac{\partial^2 g(\overline{k},\overline{\theta})}{\partial k^2}\right)\cdot\overline{k} = 0,\end{aligned} \tag{17}$$

$$\overline{c}\xi_{ck} + \overline{k}\xi_{kk} - \left(1 - d + \overline{\theta}f'(\overline{k})\right)\cdot\overline{k} = 0, \tag{18}$$

$$\overline{c}\xi_{c\theta} + \overline{k}\xi_{k\theta} - \overline{\theta}f(\overline{k}) = 0. \tag{19}$$

In the case of standard geometric discounting $(\beta = 1)$, the steady state values of $\overline{c}$ and $\overline{k}$ are uniquely determined by (11), (12) and the coefficients ξ_{kk}, $\xi_{k\theta}$, ξ_{ck} and $\xi_{c\theta}$ are identified by $(16) - (19)$. In general, there are two different eigenvalues (solutions for) ξ_{kk}. Under the standard parameterization, such a model is saddle path stable, i.e., both eigenvalues ξ_{kk} are real, and one of them is between zero and one. Choosing the latter stable solution guarantees a convergence to the steady state. Given ξ_{kk}, the remaining coefficients $\xi_{k\theta}$, ξ_{ck} and $\xi_{c\theta}$ are determined uniquely.

We next focus on the case of quasi-geometric discounting $(\beta \neq 1)$. Now, equations (11), (12) do not allow to compute the steady state values of $\overline{c}$ and $\overline{k}$ because the derivative $\frac{\partial g(\overline{k},\overline{\theta})}{\partial k}$ is unknown. Similarly, equations $(16) - (19)$ do not allow to compute the coefficients ξ_{kk}, $\xi_{k\theta}$, ξ_{ck} and $\xi_{c\theta}$ because the derivatives $\frac{\partial^2 g(\overline{k},\overline{\theta})}{\partial k^2}$ and $\frac{\partial^2 g(\overline{k},\overline{\theta})}{\partial k\partial\theta}$ are unknown. Alternatively, if we consider the system of equations (11), (12), (16) − (19), we can see that it has six equations and eight unknowns, $\overline{c}$, $\overline{k}$, ξ_{kk}, $\xi_{k\theta}$, ξ_{ck}, $\xi_{c\theta}$, $\frac{\partial^2 g(\overline{k},\overline{\theta})}{\partial k^2}$ and $\frac{\partial^2 g(\overline{k},\overline{\theta})}{\partial k \partial \theta}$, and thus, the solution to it is not uniquely determined. We also observe that considering higher-order approximations does not help to identify the steady state and the near-steady-state dynamics because higher-order derivatives of the policy function are also unknown. For example, under the second-order approximation, the unknown derivatives are $\frac{\partial^3 g(\overline{k},\overline{\theta})}{\partial k^3}$, $\frac{\partial^3 g(\overline{k},\overline{\theta})}{\partial k^2 \partial \theta}$, $\frac{\partial^3 g(\overline{k},\overline{\theta})}{\partial k \partial \theta^2}$.

Our assumption that the model is consistent with the steady state implies that the eigenvalue satisfies $\xi_{kk} \in (0,1)$.[4] By using equation (11), we obtain that the corresponding interval for the capital stock is

$$\overline{k} \in \left((f')^{-1} \left(\frac{1-(1-d)\,\beta\delta}{\overline{\theta}\beta\delta} \right),\ (f')^{-1} \left(\frac{1-\delta\,(1-d\beta)}{\overline{\theta}\beta\delta} \right) \right). \tag{20}$$

Given that the solution to (11), (12), (16) − (19) is not identified, each value of capital within this interval can be a potential steady state.

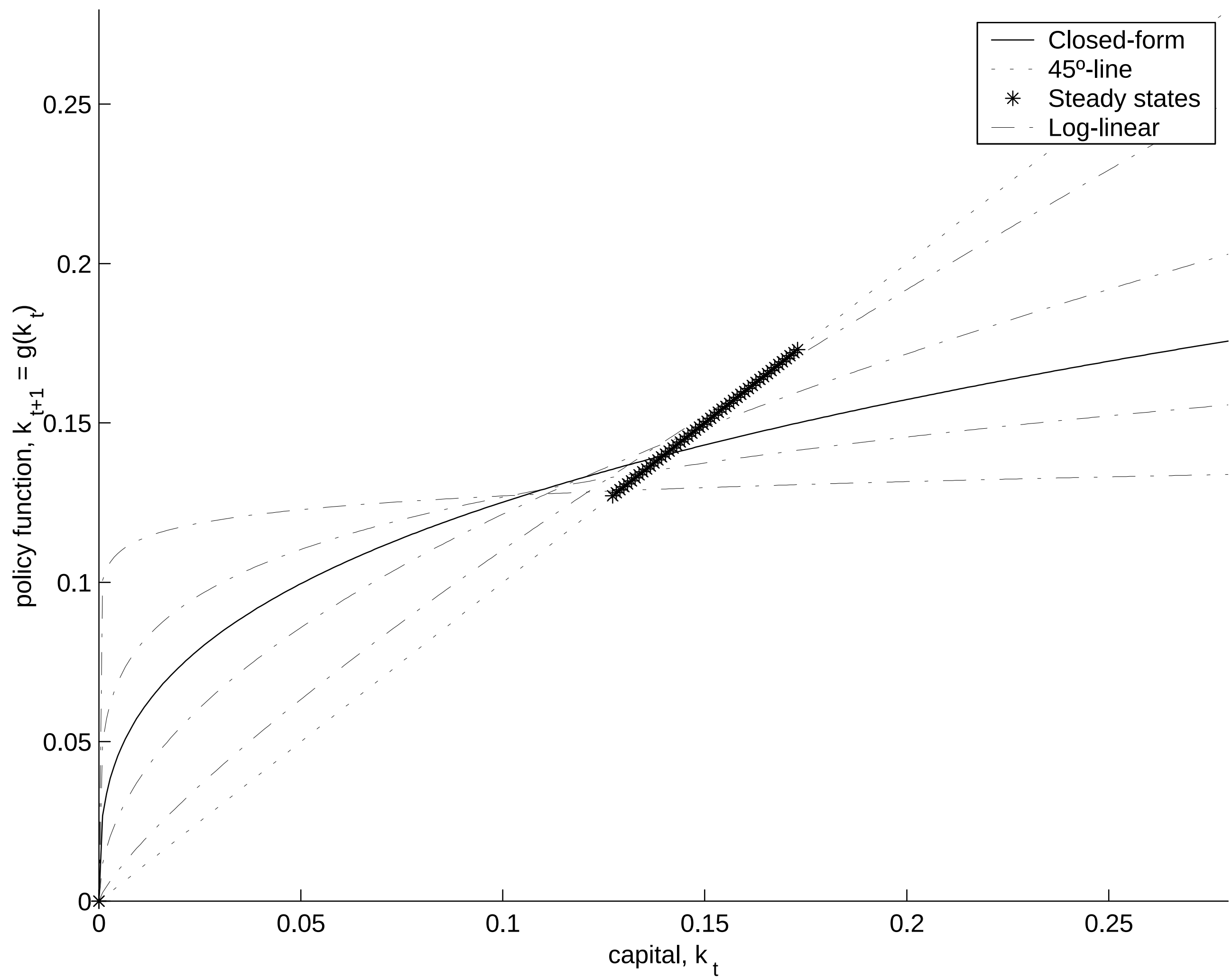
Let us consider first the log-linearized version of the deterministic model. Here, we have that $\xi_{k\theta} = 0$, $\xi_{c\theta} = 0$ and that the near-steady-state dynamics are described by two equations, (16) and (18). Observe that if a value of ξ_{kk} is fixed, then (11), (12), (16), (18) uniquely determine $\overline{k}$, $\overline{c}$, $\frac{d^2 g(\overline{k})}{dk^2}$ and ξ_{ck}. Hence, for each $\xi_{kk} \in (0,1)$, there exist a unique steady state and a unique policy function, consistent with this particular steady state. As an illustration, in Figure 1, we plot the set of steady states and several examples of the log-linear policy functions for the model with the closed-form solution parametrized by $\beta = 0.8$ (here and further in this section, we assume that $\delta = 0.96$ and $\alpha = 0.33$). The slope of the constructed policy functions, ξ_{kk}, ranges from zero to one in the steady states with the lowest and highest values of capital, respectively. The log-linear approximation with $\xi_{kk} = \alpha$ coincides with the closed-form solution.

6 Optimality of the log-linear policy rules

To establish the optimality of the constructed log-linear policy functions, we compare the implied lifetime utilities. We specifically address the following

[4]Under all parametrizations considered, we observe that if one eigenvalue is fixed in the interval $(0,1)$, then the other eigenvalue is real and strictly larger than one.

Figure 1. Case β = 0.8: Closed-form solution, log-linear solutions and the steady states.



question is: "Does the closed-form solution give a higher lifetime utility than do the other constructed solutions?" Below, we elaborate an example, which shows that the closed-form solution is not always the best alternative. Assume that initial capital k_0 is in interval (20). Denote by $W^{\overline{k}}(k_0)$ the lifetime utility of the agent who, given initial capital k_0, makes all choices according to a log-linear policy rule leading to a steady state $\overline{k}$. Consider the utility difference, $\Delta W \equiv W^{\overline{k}^*}(k_0) - W^{k_0}(k_0)$, where $\overline{k}^*$ is the steady state of the closed-form solution. In other words, we compare the lifetime utility derived from moving to the steady state of the closed-form solution, $W^{\overline{k}^*}(k_0)$, with the one obtained from maintaining initial capital forever, $W^{k_0}(k_0)$, (note that in the absence of uncertainty, the choice $k_{t+1} = k_0$ for all t is optimal according to the log-linear policy function leading to k_0). In Figure 2, we plot ΔW as a function of k_0 under $\beta \in \{0.8,\ 0.9\}$. As we see, the closed-form solution gives a higher (lower) lifetime utility than solutions leading to k_0, for all k_0, which are lower (higher) than $\overline{k}^*$.

We next ask: "Is it true that a solution, whose steady state is lower than $\overline{k}^*$, always gives a lower lifetime utility than the closed-form solution does?" In the model with $\beta = 0.8$, we compute the difference between lifetime utilities under the closed-form and log-linear solutions, $\Delta W = W^{\overline{k}^*}(k_0) - W^{\overline{k}}(k_0)$, where $\overline{k} = 0.99\overline{k}^*$. The results are shown in Figure 3. As we see, for very low values of k_0, the log-linear solution with the steady state $\overline{k}$ dominates in utility the closed-form solution. Our general conclusion is therefore that the ranking of multiple equilibria in the log-linearized version of the model depends on a specific initial condition.

We shall now consider the log-linearized model with uncertainty. Again, if $\xi_{kk} \in (0,1)$ is fixed, then equations (11), (12) uniquely determine the corresponding steady state values of $\overline{k}$ and $\overline{c}$, and equations (16), (18) determine $\frac{\partial^2 g(\overline{k},\overline{\theta})}{\partial k^2}$ and ξ_{ck}, respectively. However, the remaining two equations (17), (19) are not sufficient to determine three unknowns $\xi_{k\theta}$, $\xi_{c\theta}$ and $\frac{\partial^2 g(\overline{k},\overline{\theta})}{\partial k \partial \theta}$. Therefore, in the log-linearized model with uncertainty the problem of indeterminacy is even more severe than in the deterministic one. Specifically, there exists a continuum of log-linear solutions leading to each possible steady state.

We finally investigate how the introduction of uncertainty affects the ranking of the solutions. Let us assume that $\theta_0 = \overline{\theta} = 1$. As shown in Appendix B, under the log-linear solution, the expected lifetime utility in the model with uncertainty is equal to the lifetime utility in the determin-

Figure 2. Utility difference under the closed-form and other log-linear solutions.

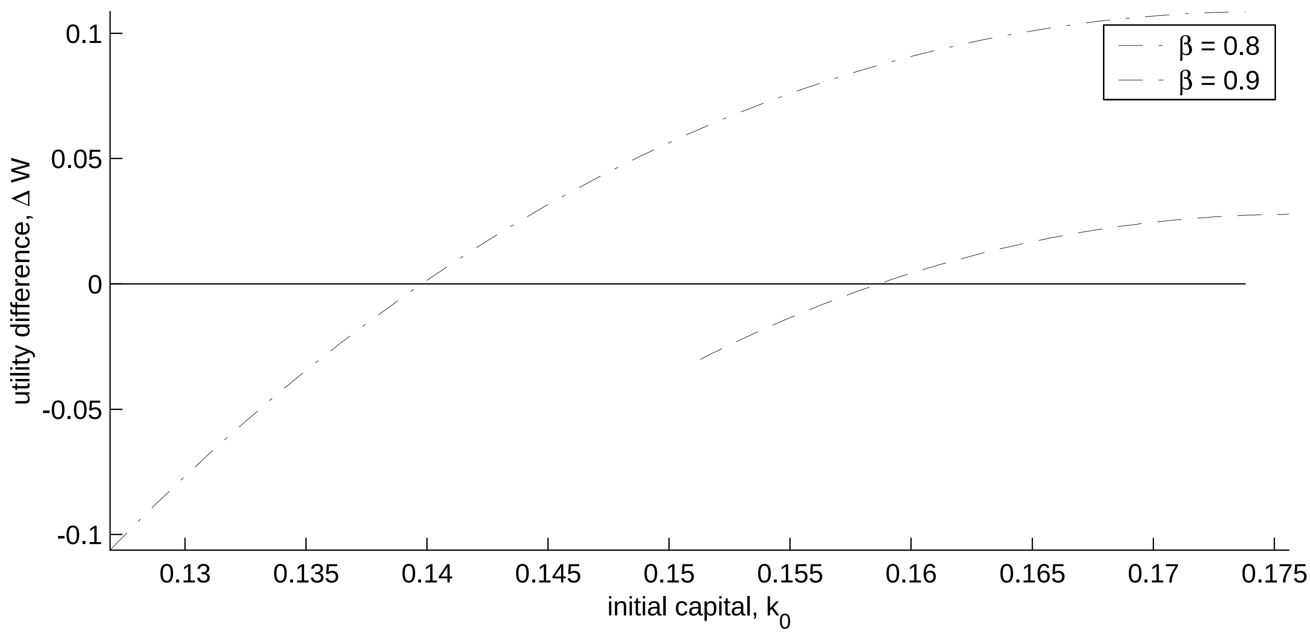
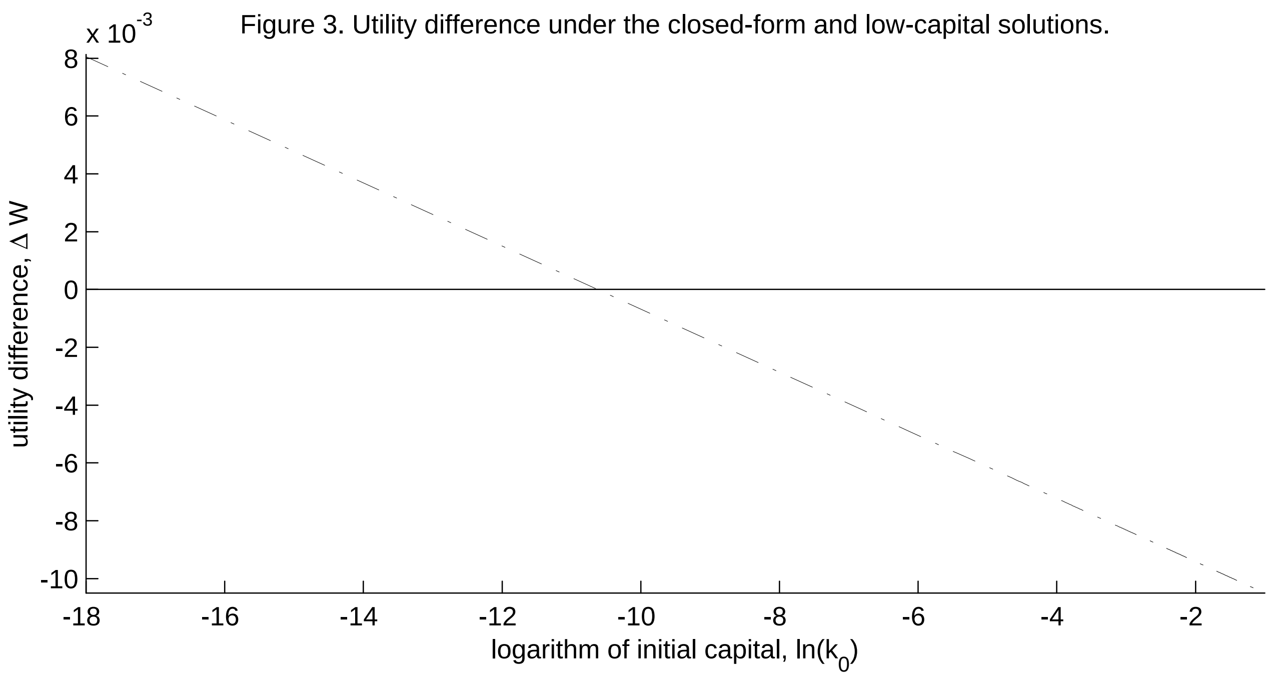


Figure 3. Utility difference under the closed-form and low-capital solutions.



istic model, $W^{\overline{k}}(k_0, \theta_0) = W^{\overline{k}}(k_0)$. Therefore, Figures 2 and 3, as well as the previous discussion of the deterministic case, apply to the stochastic case without modifications.

7 Discussion

In this section, we compare our results to those in Krusell and Smith (2000). For the deterministic version of the model, $\theta_t = \overline{\theta}$ for all t, $g\left(k_t, \overline{\theta}\right) \equiv g(k_t)$, with full depreciation of capital, $d = 1$, Krusell and Smith (2000) show that each value of the capital stock in interval (20) can be a steady state. It is argued that the equilibria are expectation-driven: a particular steady-state that prevails depends upon the agent's optimism (pessimism) about her future saving behavior. Furthermore, it is shown that there exist infinitely many discontinuous policy rules in the form of step functions leading to each steady state. Krusell and Smith (2000) also demonstrate that at least for some initial conditions, the constructed step-function solutions can give a higher level of the lifetime utility than does the closed-form solution.

The results we have for the log-linearized version of the model are in many respects similar. The log-linearized equilibrium conditions do not allow us to determine the equilibrium value of ξ_{kk}. Assuming that the model is consistent with the steady state, we have that $\xi_{kk} \in (0,1)$ and thus, each value of capital in interval (20) can be a steady state. In our case, the agent's beliefs about the future also predetermine the equilibrium, which can be seen from the steady state expression of the Euler equation (11). The agent's decisions at time t depend on the steady state marginal propensity to save out of capital, $\xi_{kk} = \frac{\partial g\left(\overline{k}, \overline{\theta}\right)}{\partial k}$. If $\beta < 1$, it appears as if an optimistic agent (i.e., the one who expects herself to save much in the future) faces a higher return on capital than the pessimistic one. If $\beta > 1$, the situation reverses. Regarding the number of paths leading to each steady state, in the deterministic case, we have a unique log-linear decision rule associated with each steady state, and in the stochastic case, we have a continuum of the log-linear decision rules corresponding to each steady state. Finally, we show that the closed-form and the constructed log-linear solutions cannot be ranked across the entire state space, which is the result parallel to the one shown in Krusell and Smith (2000) for the step-function solutions.

We should emphasize an important difference between our results and those in Krusell and Smith (2000). The indeterminacy and multiplicity of equilibria they describe are generic properties of a model with quasi-geometric discounting. The indeterminacy and multiplicity of equilibria we encounter are not the properties of the model with quasi-geometric discounting but the outcome of log-linearization. As the example with the closed-form solution shows, the original non-linear model has a unique interior solution.

8 Conclusion

In this paper, we investigate the properties of solutions to a log-linearized version of the neoclassical growth model with quasi-geometric discounting. The two main implications of our analysis are as follows:

First, the log-linearization method, which allows us to easily find an interior solution in the standard geometric discounting case, cannot be used if discounting is quasi-geometric. The log-linearized version of the model with quasi-geometric consumer has multiple solutions even though the original non-linear model has a unique interior solution. Thus, if our objective is to find an interior solution, we should apply non-linear Euler equation methods, such as the perturbation algorithm developed in Krusell, Kuruşçu and Smith (2001) or the grid- and simulation-based parameterized expectations algorithms described in Maliar and Maliar (2002).

Second, if discounting is quasi-geometric, there is a conceptual problem with the assumption of an interior solution in the sense that it does not in general guarantee the maximum level of utility. In the context of the model with a closed-form solution, Krusell and Smith (2000) show that the step-function decision rules can give higher levels of lifetime utility than does the closed-form (interior) solution. In this paper, we show that there also exist smooth (log-linear) decision rules that can dominate in the utility levels the closed-form solution. Possibly, further research on equilibrium refinement will provide a justification for the assumption of interior solution.

References

[1] Barro, R. , 1999. Ramsey meets Laibson in the neoclassical growth model, Quarterly Journal of Economics 114 (4), 1125-1152.

[2] Caillaud, B., Jullien, B., 2000. Modelling time-inconsistent preferences, European Economic Review 44, 1116-1124.

[3] Harris, C., Laibson, D., 2001. Dynamic choices of hyperbolic consumers, Econometrica 69 (4), 935-959.

[4] Krusell, P., Smith, A. , 2000. Consumption-savings decisions with quasi-geometric discounting, CEPR discussion paper no. 2651.

[5] Krusell, P., Kuruşçu, B., Smith, A., 2001. Equilibrium welfare and government policy with quasi-geometric discounting, Manuscript, forthcoming in Journal of Economic Theory.

[6] Laibson, D., 1997. Golden eggs and hyperbolic discounting, Quarterly Journal of Economics 112(2), 443-477.

[7] Laibson, D., Repetto, A., Tobacman, J., 1998. Self-control and saving for retirement, Brookings Papers on Economic Activity 1, 91-172.

[8] Maliar, L., Maliar, S., 2002. Solving the neoclassical growth model with quasi-geometric discounting: non-linear euler-equation methods, Manuscript.

[9] Manuelli, R., Sargent, T., 1987. Exercises in dynamic macroeconomic theory, (Harvard University Press, Cambridge, Massachusetts and London, England) 1-5.

[10] Uhlig, H., 1999. A toolkit for analyzing non-linear dynamic stochastic models easily, in: Marimon, R., Scott, A., eds., Computational methods for study of dynamic economies, (Oxford University Press, New York) 30-61.

9 Appendices

In Appendix A, we derive the closed-form solution by iterating on value function "by hand". In Appendix B, we compare the lifetime utility levels in the deterministic and stochastic versions of the model.

9.1 Appendix A

In this section, we show that if the value functions V, W are continuously differentiable and if the solution satisfies the first-order conditions, then the problem of finding the policy and value functions in the model with the closed-form solution is a contraction mapping.

The proof is parallel to the one for the standard geometric discounting case (see, Manuelli and Sargent, 1987). We assume that the capital stock can take any value in the interval $k \in [0, K]$, where K is a maximum sustainable capital. This assumption insures that the optimal allocation is interior, i.e., satisfies the first-order conditions. Denote by V_j and W_j the value functions V and W on the $j-$th iteration, $j = 1, 2, ...n$. Successive approximations of the value functions V and W are obtained by iterating on the following mappings

$$W_j(k, \theta) = \max_{k'} \left\{ \ln(\theta k^\alpha - k') + \beta\delta E V_{j-1}(k', \theta') \right\}, \tag{21}$$

$$V_j(k, \theta) = \ln(\theta k^\alpha - k') + \delta E V_{j-1}(k', \theta'), \tag{22}$$

where the last formula holds for all $t > 0$. The steps of the iterative procedure are as follows:

Iteration 1. Condition (21) implies

$$W_1(k, \theta) = \max_{k'} \left\{ \ln(\theta k^\alpha - k') + \beta\delta E V_0(k', \theta') \right\}.$$

As an initial guess, we take $V_0 = 0$. This maximization problem has a corner solution: $k' = 0$. Therefore, by (22), we have $V_1(k, \theta) = \ln\theta + \alpha \ln k$.

Iteration 2. Equation (21) becomes

$$W_2(k, \theta) = \max_{k'} \left\{ \ln(\theta k^\alpha - k') + \beta\delta E V_1(k', \theta') \right\},$$

where $V_1(k', \theta') = \ln\theta' + \alpha \ln k'$. Finding the first-order condition with respect to k' and substituting it into the agent's budget constraint gives

$$k' = \frac{\beta\delta\alpha}{1 + \beta\delta\alpha}\theta k^\alpha, \qquad c = \frac{1}{1 + \beta\delta\alpha}\theta k^\alpha. \tag{23}$$

By substituting $V_1(k', \theta')$, the law of motion for $\ln\theta'$ and also, k' and c from (23) into (22), we obtain

$$\begin{aligned} V_2(k, \theta) \quad = \quad & \ln\frac{1}{1 + \beta\delta\alpha} + \delta\alpha \ln\frac{\beta\delta\alpha}{1 + \beta\delta\alpha} + \alpha(1 + \delta\alpha)\ln k + \\ & \ln\theta + \delta\rho\ln\theta + \delta\alpha\ln\theta, \end{aligned}$$

or $V_2(k,\theta) = V_2^0 + V_2^1 \ln k + V_2^2 \ln\theta$.

Iteration 3. By following the same steps as before, we get

$$\begin{aligned} V_3(k,\theta) &= \ln\frac{1}{1+\beta\delta\alpha(1+\delta\alpha)} + \delta\ln\frac{1}{1+\beta\delta\alpha} + \\ &\quad \delta\alpha(1+\delta\alpha)\ln\frac{\beta\delta\alpha(1+\delta\alpha)}{1+\beta\delta\alpha(1+\delta\alpha)} + \delta^2\alpha\ln\frac{\beta\delta\alpha}{1+\beta\delta\alpha} + \\ &\quad \alpha\left(1+\delta\alpha+\delta^2\alpha^2\right)\ln k + \\ &\quad \ln\theta + \delta\alpha(1+\delta\alpha)\ln\theta + \delta\rho\ln\theta + \delta\rho\delta\alpha\ln\theta + \delta^2\rho\alpha\ln\theta, \end{aligned}$$

or $V_3(k,\theta) = V_3^0 + V_3^1 \ln k + V_3^2 \ln\theta$.

Iteration 4. We proceed in the same manner and obtain

$$\begin{aligned} V_4(k,\theta) &= \ln\frac{1}{1+\beta\delta\alpha\left(1+\delta\alpha+\delta^2\alpha^2\right)} + \delta\ln\frac{1}{1+\beta\delta\alpha(1+\delta\alpha)} + \\ &\quad \delta^2\ln\frac{1}{1+\beta\delta\alpha} + \delta\alpha\left(1+\delta\alpha+\delta^2\alpha^2\right)\ln\frac{\beta\delta\alpha\left(1+\delta\alpha+\delta^2\alpha^2\right)}{1+\beta\delta\alpha\left(1+\delta\alpha+\delta^2\alpha^2\right)} + \\ &\quad \delta^2\alpha(1+\delta\alpha)\ln\frac{\beta\delta\alpha(1+\delta\alpha)}{1+\beta\delta\alpha(1+\delta\alpha)} + \delta^3\alpha\ln\frac{\beta\delta\alpha}{1+\beta\delta\alpha} \\ &\quad \alpha\left(1+\delta\alpha+\delta^2\alpha^2+\delta^3\alpha^3\right)\ln k + \\ &\quad \ln\theta + \delta\alpha\left(1+\delta\alpha+\delta^2\alpha^2\right)\ln\theta + \delta\rho\ln\theta + \delta\rho\delta\alpha(1+\delta\alpha)\ln\theta + \\ &\quad \delta^3\rho^3\ln\theta + \delta^2\rho^2\delta\alpha\ln\theta, \end{aligned}$$

or $V_4(k,\theta) = V_4^0 + V_4^1 \ln k + V_4^2 \ln\theta$.

Generation of the Conjecture. Until now, we have obtained four elements of the sequence of value functions $\{V_j(k,\theta)\}_{j=1}^4$. Note that all functions in this sequence have the form $V_j(k,\theta) = V_j^0 + V_j^1 \ln k + V_j^2 \ln\theta$. We show now that the sequence of the value functions converges to the value function $V(k,\theta) = V^0 + V^1 \ln k + V^2 \ln\theta$, as $j \to \infty$, where $V^i = \lim_{j\to\infty} V_j^i$, $i = 0,1,2$. To find the limits, we use the algebra of geometric series.

The first term V^0 is given by

$$V^0 = \lim_{j\to\infty} V_j^0 = \lim_{j\to\infty} \sum_{t=0}^{j-2} \delta^t \ln \left(\frac{1}{1+\beta\delta\alpha\left(1+\delta\alpha+...+\delta^{j-t-2}\alpha^{j-t-2}\right)} \right) +$$

$$\lim_{j\to\infty} \sum_{t=0}^{j-2} \delta^t \left\{ \delta\alpha\left(1+\delta\alpha+...+\delta^{j-t-2}\alpha^{j-t-2}\right) \times \right.$$

$$\left. \ln \left(\frac{\beta\delta\alpha\left(1+\delta\alpha+...+\delta^{j-t-2}\alpha^{j-t-2}\right)}{1+\beta\delta\alpha\left(1+\delta\alpha+...+\delta^{j-t-2}\alpha^{j-t-2}\right)} \right) \right\} =$$

$$(1-\delta)^{-1} \ln \frac{1-\delta\alpha}{1-\delta\alpha+\beta\delta\alpha} + (1-\delta)^{-1} \frac{\delta\alpha}{1-\delta\alpha} \ln \frac{\beta\delta\alpha}{1-\delta\alpha+\beta\delta\alpha}.$$

The second term V^1 is

$$V^1 = \lim_{j\to\infty} V_j^1 = \lim_{j\to\infty} \alpha\left(1+\delta\alpha+...+\delta^{j-1}\alpha^{j-1}\right) = \frac{\alpha}{1-\delta\alpha}.$$

The last term V^2 can be found as

$$V^2 = \lim_{j\to\infty} V_j^2 = \lim_{j\to\infty} \left(1+\delta\rho+...+\delta^{j-1}\rho^{j-1}\right) \ln\theta +$$

$$\lim_{j\to\infty} \sum_{t=0}^{j-2} \delta^t \rho^t \delta\alpha \left(1+\delta\rho+...+\delta^{j-2-t}\rho^{j-2-t}\right) \ln\theta =$$

$$\left(\frac{1}{1-\delta\rho} + \frac{\delta\alpha}{(1-\delta\rho)(1-\delta\alpha)} \right) \ln\theta = \frac{1}{(1-\delta\rho)(1-\delta\alpha)} \ln\theta.$$

Therefore, the constructed sequence of value functions converges to (9).

9.2 Appendix B

In this section, we derive the expected lifetime utility under the log-linear solution and show that if $\theta_0 = \overline{\theta} = 1$, then the expected lifetime utility in the stochastic economy is equal to the lifetime utility in the deterministic economy, i.e., $W^{\overline{k}}(k_0, 1) = W^{\overline{k}}(k_0)$.

The log-linear policy functions for consumption and capital in (15) can be written as

$$\frac{c_t}{\overline{c}} = \left(\frac{k_t}{\overline{k}}\right)^{\xi_{kc}} \left(\frac{\theta_t}{\overline{\theta}}\right)^{\xi_{c\theta}}, \qquad \frac{k_{t+1}}{\overline{k}} = \left(\frac{k_t}{\overline{k}}\right)^{\xi_{kk}} \left(\frac{\theta_t}{\overline{\theta}}\right)^{\xi_{k\theta}}.$$

Substituting recursively $\frac{k_t}{\overline{k}}$ into $\frac{c_t}{\overline{c}}$ yields

$$\frac{c_t}{\overline{c}} = \left(\frac{k_0}{\overline{k}}\right)^{\xi_{kk}^t \cdot \xi_{kc}} \left(\frac{\theta_t}{\overline{\theta}}\right)^{\xi_{c\theta}} \prod_{i=1}^{t} \left(\frac{\theta_{t-i}}{\overline{\theta}}\right)^{\xi_{kk}^{i-1} \cdot \xi_{k\theta} \cdot \xi_{kc}}. \tag{24}$$

The expected lifetime utility is given by

$$W^{\overline{k}}(k_0, \theta_0) = \ln(c_0) + \beta E_0 \left\{ \delta \ln(c_1) + \delta^2 \ln(c_2) + ... \right\}. \tag{25}$$

Equation (24) and the law of motion for shock, $\ln \theta_{t+1} = \rho \ln \theta_t + \varepsilon_{t+1}$, where $0 \leq \rho < 1$ and $\varepsilon_{t+1} \sim N(0, \sigma^2)$, imply

$$E_0 \{\ln(c_t)\} = \ln(\overline{c}) + \xi_{kk}^t \xi_{kc} \ln\left(\frac{k_0}{\overline{k}}\right) + \xi_{c\theta}\left(\rho^t \ln(\theta_0) - \ln(\overline{\theta})\right) + \sum_{i=1}^{t} \xi_{kk}^{i-1} \xi_{k\theta} \xi_{kc} \left(\rho^{t-i} \ln(\theta_0) - \ln(\overline{\theta})\right).$$

Then, substituting the last result into (25) for all t yields

$$W^{\overline{k}}(k_0, \theta_0) = \left(1 + \frac{\beta\delta}{1-\delta}\right) \ln(\overline{c}) + \xi_{kc}\left(\ln(k_0) - \ln(\overline{k})\right)\left(1 + \frac{\beta\delta\xi_{kk}}{1-\delta\xi_{kk}}\right) + \left(\xi_{c\theta} + \frac{\xi_{k\theta}\xi_{kc}\beta\delta}{1-\delta\xi_{kk}}\right)\left(\left(1 + \frac{\beta\delta\rho}{1-\delta\rho}\right)\ln(\theta_0) - \left(1 + \frac{\beta\delta}{1-\delta}\right)\ln(\overline{\theta})\right).$$

Assume that $\theta_0 = \overline{\theta} = 1$. Then, the expected lifetime utility in the stochastic economy is equal to the lifetime utility in the deterministic economy, $W^{\overline{k}}(k_0, \theta_0) = W^{\overline{k}}(k_0)$.